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A sufficient condition for a finite group to be a Borsuk-Ulam group

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1 Introduction

In this paper, we always assume that a group means a finite group. A G-map $f\colon X \to Y$ is said to be a G-isovariant map if $G_x = G_{f(x)}$ for any $x \in X$, where G_x is the isotropy subgroup, that is, $G_x = \{g \in G \mid g \cdot x = x\}$. We call a group G is a BUG (Borsuk-Ulam group) [5] if

$$\dim V - \dim V^G \le \dim W - \dim W^G$$

for any isovariant G-map $f\colon V \to W$ between G-representation spaces V and W.

Let C_2 be a cyclic group of order 2 and let $f\colon V \to W$ be an isovariant C_2-map between C_2-representation spaces V and W. Fixing a G-invariant inner product, f induces a free C_2-map $S(f|_{V-V^{C_2}})\colon S(V - V^{C_2}) \to S(W - W^{C_2})$ between C_2-representation spheres, where $V - V^{C_2}$ is an orthogonal vector subspace of V^{C_2} in V. By Borsuk-Ulam theorem, this map gives $\dim S(V - V^{C_2}) \le \dim S(W - W^{C_2})$. Since $\dim S(V - V^{C_2}) = \dim V - \dim V^{C^2} - 1$, C_2 is a BUG. For a cyclic group C_p of prime order p, Kobayashi [2] showed that $\dim S(V) \le \dim S(W)$ for a free C_p-map $S(f')\colon S(V) \to S(W)$ between representation spheres and thus C_p is a BUG.

Let G be a group extension of K by H: $1 \to H \to G \to K \to 1$ and $f\colon V \to W$ be an isovariant G-map. Since the equality

$$\begin{aligned}\dim W - \dim W^G\ &-(\dim V - \dim V^G)\\ &= (\dim W - \dim W^H - (\dim V - \dim V^H))\\ &\quad + (\dim W^H - \dim W^G - (\dim V^H - \dim V^G))\end{aligned}$$

holds, if K and H are BUGs then G is a BUG [5]. Therefore any solvable group is a BUG. Then it is natural to ask whether a group is a BUG or not.

Wasserman [5] proposed a prime condition which implies a sufficient condition for a group to be a BUG. A positive integer n satisfies the *prime condition* if

$$p_1^{-1} + p_2^{-1} + \cdots + p_r^{-1} < 1,$$

where $p_1, \ldots, p_r$ are primes and $e_1, \ldots, e_r$ are positive integers such that $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. A group G satisfies the *prime condition* if the order of any cyclic subgroup of G satisfies the prime condition.

Theorem 1 ([5]) *If a group G satisfies the prime condition, then G is a BUG.*

Let $\mathrm{Cycl}(G)$ be the set of all cyclic subgroups of G. Nagasaki and Ushitaki [3] proposed a Möbius condition: A group G satisfies the *Möbius condition* if

$$\sum_{\substack{D \in \mathrm{Cycl}(G) \\ C \leq D}} \mu\left(\frac{|D|}{|C|}\right) \geq 0$$

for any cyclic subgroup C of G, where $\mu\colon \mathbb{N} \to \{0, \pm 1\}$ is the Möbius function, that is,

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^r & n = p_1 p_2 \cdots p_r \text{ for distinct primes } p_1, p_2, \ldots, p_r. \end{cases}.$$

Theorem 2 ([3]) *If a group G satisfies the Möbius condition, then G is a BUG.*

Since if K and H are BUGs then a group extension of H by K is a BUG, if we obtain that every simple group is a BUG, then any group is a BUG. By the above theorem, Nagasaki and Ushitaki showed that projective linear groups $\mathrm{PSL}(2, q)$ are BUGs. In this paper, we give a sufficient condition for a group to be a BUG and apply projective linear groups $\mathrm{PSL}(3, q)$ and alternating groups A_n.

2 A sufficient condition

Let V and W be G-representation spaces and let $f\colon V \to W$ be an isovariant G-map. For a subgroup H of G, let

$$g_f(H) = (\dim W - \dim W^H) - (\dim V - \dim V^H).$$

Note that If G is a cyclic group then $g_f(G) \geq 0$.

Proposition 3 *Let H_1 and H_2 be a subgroups of G with $H_1 \triangleleft H_2$ and f an isovariant G-map between representation spaces.*

$$g_f(H_2) - g_f(H_1) = g_{f^{H_1}}(H_2/H_1)$$

holds. In particular, if H_2/H_1 is a BUG, *$g_f(H_2) \geq g_f(H_1)$ holds.*

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. It is made into a poset by defining $H \leq K$ in $\mathcal{S}(G)$ if H is a subgroup of K. Let $\mathrm{Cycl}(G)$ be the full subposet of $\mathcal{S}(G)$ which contains all cyclic subgroups of G.

We put

$$\mu(C, D) = \begin{cases} \mu(\frac{|D|}{|C|}), & C \leq D \\ 0, & \text{otherwise.} \end{cases}$$

Nagasaki and Ushitaki [3] showed that $\mathrm{PSL}(2, q)$ satisfies the Möbius condition by using the following equation.

Theorem 4 ([3]) *Let $f\colon V \to W$ be a G-map between representation spaces.*

$$|G|g_f(G) = \sum_{C\in \mathrm{Cycl}(G)} \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \right) |C|g_f(C)$$

holds. If G satisfies the Möbius condition then G is a BUG.

Let $\mathrm{RCycl}(G)$ be the set of representatives of conjugacy classes of all cyclic subgroups of G and let $\mathrm{RCycl}_1(G)$ be the set of representatives of conjugacy classes of all nontrivial cyclic subgroups of G. Recall that $g_f(\{e\}) = 0$.

Let

$$\tilde{\mu}(C,D) = \begin{cases} \mu(\frac{|D|}{|C|}), & (C) \le (D) \\ 0, & \text{otherwise.} \end{cases}$$

where (C) denotes the conjugacy class of C.

Lemma 5 *For $C \in \mathrm{RCycl}(G)$, $\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) = \sum_{D\in \mathrm{RCycl}(G)} \frac{|N_G(C)|}{|N_G(D)|}\tilde{\mu}(C,D)$.*

Proof Let $C < D \in \mathrm{Cycl}(C_G(C))$, $S = \{E \in \mathrm{Cycl}(G) \mid (D) = (E), E \ge C\}$ and let $f\colon N_G(C) \to S$ be a map which sends g to $g^{-1}Dg$. Let $E \in S$. Then $E = g^{-1}Dg$ for some $g \in G$. C and $g^{-1}Cg$ is a subgroup of E with same index. Since for any $k > 0$ dividing $|D|$, a subgroup of order k of the cyclic group D is unique, we see that $C = g^{-1}Cg$ and thus $g \in N_G(C)$ and $f(g) = E$. Therefore, the map f is surjective. Thus, $\#S = \frac{|N_G(C)|}{|N_G(D)|}$. If D_1 and D_2 are conjugate, then $\tilde{\mu}(C,D_1) = \tilde{\mu}(C,D_2)$. Therefore we see that

$$\begin{aligned} \sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) &= \sum_{D\in \mathrm{RCycl}(G)} \sum_{\substack{E\in \mathrm{Cycl}(G)\\ (E)=(D)}} \mu(C,E) \\ &= \sum_{D\in \mathrm{RCycl}(G)} \sum_{E\in S} \mu(C,D) \\ &= \sum_{D\in \mathrm{RCycl}(G)} \frac{|N_G(C)|}{|N_G(D)|}\tilde{\mu}(C,D). \end{aligned}$$

∎

Let

$$\beta_G(C,D) = \frac{|C|\tilde{\mu}(C,D)}{|N_G(D)|}$$

and

$$\beta_G(C) = \sum_{D\in \mathrm{RCycl}(G)} \beta_G(C,D).$$

We abbreviate to write $g_f(G)$ as $g(G)$ if f is clear.

Proposition 6

$$g(G) = \sum_{C\in \mathrm{RCycl}(G)} \beta_G(C)g(C). \tag{1}$$

Proof By Theorem 4 and Lemma 5, we see that

$$\begin{aligned} g(G) &= \tfrac{1}{|G|} \sum_{C \in \mathrm{Cycl}(G)} \left(\sum_{D \in \mathrm{Cycl}(G)} \mu(C,D) \right) |C| g(C) \\ &= \tfrac{1}{|G|} \sum_{C \in \mathrm{RCycl}(G)} \sum_{\substack{C' \in \mathrm{Cycl}(G) \\ (C)=(C')}} \left(\sum_{D \in \mathrm{Cycl}(G)} \mu(C',D) \right) |C'| g(C') \\ &= \sum_{C \in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \left(\sum_{D \in \mathrm{Cycl}(G)} \mu(C,D) \right) g(C) \\ &= \sum_{C \in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \left(\sum_{D \in \mathrm{RCycl}(G)} \tfrac{|N_G(C)|}{|N_G(D)|} \tilde{\mu}(C,D) \right) g(C) \\ &= \sum_{C \in \mathrm{RCycl}(G)} \left(\sum_{D \in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(D)|} \tilde{\mu}(C,D) \right) g(C). \end{aligned}$$

∎

We write $\beta(C) = \beta_G(C)$ for simple. If G is a cyclic group, then

$$\beta(C) = \sum_{D \in \mathrm{RCycl}(G)} \frac{|C|}{|G|} \tilde{\mu}(C,D) = \begin{cases} 0, & C \neq G \\ 1, & C = G. \end{cases} \tag{2}$$

Lemma 7 $\sum_{C \in \mathrm{RCycl}(G)} \frac{1}{|N_G(C)|} \leq 1.$

Proof By the class equation for G, we see that

$$1 = \sum_{(x)} \frac{1}{|C_G(x)|} \geq \sum_{C \in \mathrm{RCycl}(G)} \frac{1}{|C_G(C)|} \geq \sum_{C \in \mathrm{RCycl}(G)} \frac{1}{|N_G(C)|}.$$

∎

Lemma 8 *$|G| = \sum_{C,D \in \mathrm{Cycl}(G)} \mu(C,D)|C|$ and $\sum_{C \in \mathrm{RCycl}(G)} \beta(C) = 1$. If G is nontrivial then $\sum_{C \in \mathrm{RCycl}_1(G)} \beta(C) > 0$.*

Proof Let $u\colon \mathrm{Cycl}(G) \to \mathbb{Q}$ be a map defined as

$$u(C) = \begin{cases} |\mathrm{gen} C| & C \neq \{e\} \\ 1 & C = \{e\} \end{cases}$$

and put $v(G) = \sum_{D \in \mathrm{Cycl}(G)} u(D)$. Then $v(G) = |G|$. By the Möbuis inversion formula, we see

$$u(D) = \sum_{C \leq D \in \mathrm{Cycl}(G)} \mu(|D|/|C|) v(C) = \sum_{C \in \mathrm{Cycl}(G)} \mu(C,D) v(C)$$

and then

$$|G| = v(G) = \sum_{C,D\in \mathrm{Cycl}(G)} \mu(C,D)|C|.$$

Therefore, we see

$$\begin{aligned}
1 &= \tfrac{1}{|G|} \sum_{C\in \mathrm{Cycl}(G)} |C| \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \right) \\
&= \tfrac{1}{|G|} \sum_{C\in \mathrm{RCycl}(G)} \sum_{\substack{C'\in \mathrm{Cycl}(G)\\ (C)=(C')}} |C'| \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C',D) \right) \\
&= \sum_{C\in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \\
&= \sum_{C\in \mathrm{RCycl}(G)} \sum_{D\in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \tfrac{|N_G(C)|}{|N_G(D)|} \tilde{\mu}(C,D) \\
&= \sum_{C\in \mathrm{RCycl}(G)} \sum_{D\in \mathrm{RCycl}(G)} \beta(C,D) \\
&= \sum_{C\in \mathrm{RCycl}(G)} \beta(C).
\end{aligned}$$

If G is nontrivial then, by Lemma 7, we see

$$\beta(\{e\}) = \sum_{D\in \mathrm{RCycl}(G)} \frac{\mu(|D|)}{|N_G(D)|} < \sum_{D\in \mathrm{RCycl}(G)} \frac{1}{|N_G(D)|} \le 1$$

and thus $\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C) > 0$. ∎

Now we consider $\sum_{C\in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C)$ for a map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\ge 0}$. We note that G satisfies the Möbius condition if and only if $\sum_{C\in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \ge 0$ for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\ge 0}$. By (2), we see

Proposition 9 *A cyclic group satisfies the Möbius condition.*

We recall Proposition 3. For an isovariant G-map f and subgroups $C \triangleleft D$ of G such that D/C is a BUG, $g_f(C) \le g_f(D)$. We say G is a CCG (cyclic condition group), if for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\ge 0}$ such that $\gamma(C) \le \gamma(D)$ if $(C) \le (D)$, $\sum_{C\in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \ge 0$. A CCG is a BUG.

Proposition 10 *A group satisfying the prime condition is a* CCG.

Proof Let $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\ge 0}$ be a map such that $\gamma(C) \le \gamma(D)$ if $(C) \le (D)$. Since

$$\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C)\gamma(C) = \sum_{D\in \mathrm{RCycl}_1(G)} \sum_{C\in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C),$$

we show that $\sum\limits_{C\in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C) \geq 0$ for each $D \in \mathrm{RCycl}_1(G)$. For an positive integer n, let $\pi(n)$ be the set of all primes dividing n. Put $r = \#\pi(|D|)$, the number of elements of $\pi(|D|)$, and let D_0 be the subgroup of D with index $\prod\limits_{p\in\pi(|D|)} p$. Since

$$\binom{r}{2s}(r-2s) = \binom{r}{2s+1}(2s+1),$$

we see that

$$\begin{aligned}
&\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C) \\
&= \sum_{\substack{C\in \mathrm{RCycl}_1(G)\\ (D_0)\leq(C)\leq(D)}} \beta(C,D)\gamma(C) \\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \left(\sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \beta(E,D)\gamma(E) + \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq(F)<(D)\\ |\pi(|D|/|F|)|=2s+1}} \beta(F,D)\gamma(F) \right) \\
&= \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \left(\beta(E,D)\gamma(E) + \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq(F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \beta(F,D)\gamma(F) \right) \\
&= \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \left(\frac{|E|}{|N_G(D)|}\gamma(E) - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq(F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \frac{|F|}{|N_G(D)|}\gamma(F) \right) \\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \left(\frac{|E|}{|N_G(D)|}\gamma(E) - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq(F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \frac{|F|}{|N_G(D)|}\gamma(E) \right) \\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \left(1 - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (F)<(E)\\ |\pi(|E|/|F|)|=1}} \frac{|F|}{|E|} \right) \frac{|E|}{|N_G(D)|}\gamma(E) \\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq(D)\\ |\pi(|D|/|E|)|=2s}} \left(1 - \sum_{p\in\pi(|E|)} \frac{1}{p} \right) \frac{|E|}{|N_G(D)|}\gamma(E)
\end{aligned}$$

and thus if G satisfies the prime condition, then it is nonnegative. ∎

3 Through linear programming

Let $\mathrm{RCycl}_1^+(G)$ and $\mathrm{RCycl}_1^-(G)$ be the subsets of $\mathrm{RCycl}_1(G)$ consisting of C with $\beta_G(C) > 0$ and $\beta_G(C) < 0$, respectively. We consider a map

$$\psi\colon \mathrm{RCycl}_1^-(G) \times \mathrm{RCycl}_1^+(G) \to \mathbb{Q}_{\leq 0}$$

such that $\beta_G(C) = \sum_{D \in \mathrm{RCycl}_1^+(G)} \psi(C, D)$ for $C \in \mathrm{RCycl}_1^-(G)$ and if C is not subconjugate to D then $\psi(C, D) = 0$. Then

$$\begin{aligned}
g(G) &= \sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C) g(C) \\
&= \sum_{C \in \mathrm{RCycl}_1^+(G)} \beta_G(C) g(C) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \beta_G(C) g(C) \\
&= \sum_{D \in \mathrm{RCycl}_1^+(G)} \beta_G(D) g(D) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \left(\sum_{D \in \mathrm{RCycl}_1^+(G)} \psi(C, D) \right) g(C) \\
&\geq \sum_{D \in \mathrm{RCycl}_1^+(G)} \left(\beta_G(D) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \psi(C, D) \right) g(D).
\end{aligned}$$

By Lemma 8, $\sum_{C \in \mathrm{RCycl}_1^+(G)} \beta_G(C) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \beta_G(C) > 0$. Thus we may expect the existence of ψ. We determine whether there exist ψ such that $\beta(D) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \psi(C, D) \geq 0$ for $D \in \mathrm{RCycl}_1^+(G)$ by linear programming.

$$\begin{cases}
\psi(C, D) \leq 0 \\
\psi(C, D) = 0 \text{ if } (C) \not\leq (D) \\
\sum_{D \in \mathrm{RCycl}_1^+(G)} \psi(C, D) \leq \beta_G(C) \text{ for } C \in \mathrm{RCycl}_1^-(G) \\
\sum_{C \in \mathrm{RCycl}_1^-(G)} \psi(C, D) \geq -\beta_G(D) \text{ for } D \in \mathrm{RCycl}_1^+(G)
\end{cases}$$

We can check the existence of ψ for the following groups by using the software GAP [1]:

Theorem 11 (1) *Alternating groups A_n, $5 \leq n \leq 11$ satisfy the prime condition.*

(2) *A_n, $12 \leq n \leq 21$ are* CCG*s.*

(3) *Symmetric groups S_n, $5 \leq n \leq 9$ satisfy the prime condition.*

(4) *S_n, $10 \leq n \leq 22$ are* CCG*s.*

(5) *All sporadic groups are* CCG*s.*

(6) *Automorphism groups of all sporadic groups are* CCG*s.*

4 Projective special linear group

If any simple groups are BUGs, then every group is a BUG. It is important to study simple groups. Projective special linear groups $\mathrm{PSL}(3, q)$ are simple groups.

Lemma 12 *Let C be a cyclic subgroup of a group G. Suppose that there is a unique maximal cyclic subgroup D of G with $C < D$. Then $N_G(C) = N_G(D)$, $\beta_G(C) = 0$ and $\beta_G(D) = \frac{|D|}{|N_G(D)|} > 0$.*

Proof Since $C < D$, for $g \in N_G(D)$, $g^{-1}Cg$ is a subgroup of the cyclic group D with index $|D/C|$ and thus $g^{-1}Cg = C$. Therefore, $N_G(D) \leq N_G(C)$. If $g \in N_G(C) \setminus N_G(D)$ exists, then $C < g^{-1}Dg \neq D$. This is contradiction. Therefore the equality $N_G(C) = N_G(D)$ holds.

We see that

$$\begin{aligned}\beta_G(C) &= \sum_{E \in \mathrm{RCycl}(G)} \frac{|C|}{|N_G(E)|}\tilde{\mu}(C, E) = \frac{|D|}{|N_G(D)|} \sum_{E \in \mathrm{RCycl}(D)} \frac{|C|}{|D|}\tilde{\mu}(C, E) \\ &= \frac{|D|}{|N_G(D)|}\beta_D(C) = 0\end{aligned}$$

by (2), and clearly $\beta_G(D) = \frac{|D|}{|N_G(D)|}$. ∎

Let p be a prime and q a power of p. Let C_{q-1}, C_p, and C_{q+1} be cyclic subgroups of $\mathrm{SL}(2, q)$ of order $q - 1$, p, and $q + 1$ respectively, and let $\pi \colon \mathrm{SL}(2, q) \to G$ be a natural projection. Then $\pi(C_{q\pm1})$ has order $(q \pm 1)/\gcd(q - 1, 2)$.

Proposition 13 $g(\mathrm{PSL}(2, q)) = \frac{1}{2}g(\pi(C_{q-1})) + \frac{1}{2}g(\pi(C_{q+1})) + \frac{p}{|N_{\mathrm{PSL}(2,q)}(C_p)|}g(C_p)$.

Proof We may put

$$\mathrm{RCycl}_1(\mathrm{PSL}(2, q)) = \{H \mid \{1\} < H \leq C_r, r = p, q \pm 1\}.$$

The cyclic groups $\pi(C_r)$, $r = p, q \pm 1$ are maximal and the orders are $p, (q \pm 1)/d$, respectively, which are coprime each other. Therefore, any nontrivial cyclic subgroup of G has a unique maximal cyclic subgroup of G. Thus by Lemma 12, we see

$$g(G) = \sum_{r \in \{q \pm 1, p\}} \frac{|C_r|}{|N_{\mathrm{PSL}(2,q)}(C_r)|}g(C_r).$$

∎

Nonsolvable groups $\mathrm{PSL}(2, q)$, $\mathrm{SL}(2, q)$, $\mathrm{PGL}(2, q)$, and $\mathrm{GL}(2, q)$ are all BUGs (see [3]). Furthermore, a group which does not have a simple group except $\mathrm{PSL}(2, q)$ as a subquotient group is a BUG.

Example 14 The simple group PSL$(3, 11)$ does not satisfy the prime condition, since it has an element of order 120. We confuse order (with type) with the cyclic subgroup generated by the corresponding element. For example, PSL$(3, 11)$ has a unique cyclic group of order 110 up to conjugate, and two cyclic subgroups of order 5 up to conjugate, denoted by $5a$ and $5b$, whose are not conjugate. We have

$$\begin{aligned}\mathrm{RCycl}_1^+(\mathrm{PSL}(3,11)) &= \{10b, 10c, 10d, 11a, 110, 120, 133\},\\ \mathrm{RCycl}_1^-(\mathrm{PSL}(3,11)) &= \{2, 5a, 5b, 10a, 11b\}.\end{aligned}$$

Since $\beta(133) + \beta(11b) = 587/1815 > 0$, $\beta(110) + \beta(5b) + \beta(10a) = 0$, and $\beta(10b) + \beta(2) + \beta(5a) = 0$, we see that

$$\begin{aligned}g(\mathrm{PSL}(3,11)) \geq \beta(10c)\gamma(10c) + \beta(10d)\gamma(10d) + \beta(11a)\gamma(11a)\\ +\beta(120)\gamma(120) + \tfrac{587}{1815}\gamma(133) \geq 0.\end{aligned}$$

The group PSL$(3, 11)$ is a CCG. See the following table corresponds with $\beta_{\mathrm{PSL}(3,11)}(-,-)$. The first columns and first rows are all cyclic subgroups C and D of $\mathrm{RCycl}_1(\mathrm{PSL}(3, 11))$ respectively, and the last columns are the values $\beta_{\mathrm{PSL}(3,11)}(-)$.

$C\backslash D$	1	2	3	4	$5a$	$5b$	6	7	8	$10a$
1	1/212427600	−1/13200	−1/240	0	−1/200	−1/13200	1/240	−1/399	0	1/13200
2	0	1/6600	0	−1/120	0	0	−1/120	0	0	−1/6600
3	0	0	1/80	0	0	0	−1/80	0	0	0
4	0	0	0	1/60	0	0	0	0	−1/60	0
$5a$	0	0	0	0	1/40	0	0	0	0	0
$5b$	0	0	0	0	0	1/2640	0	0	0	−1/2640
6	0	0	0	0	0	0	1/40	0	0	0
7	0	0	0	0	0	0	0	1/57	0	0
8	0	0	0	0	0	0	0	0	1/30	0
$10a$	0	0	0	0	0	0	0	0	0	1/1320
$10b$	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$10b$	$10c$	$10d$	$11a$	$11b$	12	15	19	20	22
1	1/100	1/100	1/200	−1/1210	−1/133100	0	1/240	−1/399	0	1/1100
2	−1/50	−1/50	−1/100	0	0	1/120	0	0	1/120	−1/550
3	0	0	0	0	0	0	−1/80	0	0	0
4	0	0	0	0	0	−1/60	0	0	−1/60	0
$5a$	−1/20	0	−1/40	0	0	0	0	0	0	0
$5b$	0	−1/20	0	0	0	0	−1/48	0	0	0
6	0	0	0	0	0	−1/40	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
$10a$	0	0	0	0	0	0	0	0	−1/24	0
$10b$	1/10	0	0	0	0	0	0	0	0	0
$10c$	0	1/10	0	0	0	0	0	0	0	0
$10d$	0	0	1/20	0	0	0	0	0	0	0
$11a$	0	0	0	1/110	0	0	0	0	0	0
$11b$	0	0	0	0	1/12100	0	0	0	0	−1/100
12	0	0	0	0	0	1/20	0	0	0	0
15	0	0	0	0	0	0	1/16	0	0	0
19	0	0	0	0	0	0	0	1/21	0	0
20	0	0	0	0	0	0	0	0	1/12	0
22	0	0	0	0	0	0	0	0	0	1/50
24	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	24	30	40	55	60	110	120	133	$\beta(C)$
1	0	$-1/240$	0	$1/1100$	0	$-1/1100$	0	$1/399$	$127/7260$
2	0	$1/120$	0	0	$-1/120$	$1/550$	0	0	$-1/20$
3	0	$1/80$	0	0	0	0	0	0	0
4	$1/60$	0	$1/60$	0	$1/60$	0	$-1/60$	0	0
$5a$	0	0	0	0	0	0	0	0	$-1/20$
$5b$	0	$1/48$	0	$-1/220$	0	$1/220$	0	0	$-1/20$
6	0	$-1/40$	0	0	$1/40$	0	0	0	0
7	0	0	0	0	0	0	0	$-1/57$	0
8	$-1/30$	0	$-1/30$	0	0	0	$1/30$	0	0
$10a$	0	$-1/24$	0	0	$1/24$	$-1/110$	0	0	$-1/20$
$10b$	0	0	0	0	0	0	0	0	$1/10$
$10c$	0	0	0	0	0	0	0	0	$1/10$
$10d$	0	0	0	0	0	0	0	0	$1/20$
$11a$	0	0	0	0	0	0	0	0	$1/110$
$11b$	0	0	0	$-1/100$	0	$1/100$	0	0	$-6/605$
12	$-1/20$	0	0	0	$-1/20$	0	$1/20$	0	0
15	0	$-1/16$	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	$-1/21$	0
20	0	0	$-1/12$	0	$-1/12$	0	$1/12$	0	0
22	0	0	0	0	0	$-1/50$	0	0	0
24	$1/10$	0	0	0	0	0	$-1/10$	0	0
30	0	$1/8$	0	0	$-1/8$	0	0	0	0
40	0	0	$1/6$	0	0	0	$-1/6$	0	0
55	0	0	0	$1/20$	0	$-1/20$	0	0	0
60	0	0	0	0	$1/4$	0	$-1/4$	0	0
110	0	0	0	0	0	$1/10$	0	0	$1/10$
120	0	0	0	0	0	0	$1/2$	0	$1/2$
133	0	0	0	0	0	0	0	$1/3$	$1/3$

Table 1: $\beta_{\mathrm{PSL}(3,11)}(-,-)$

The group $\mathrm{SL}(3,q)$ is of order $q^3(q-1)^2(q+1)(q^2+q+1)$. Put $q=p^u$, $G=\mathrm{SL}(3,q)$, $\delta=1$, $d=\gcd(3,q-\delta)$, $\rho^r=1$, $r=q-\delta$, $r'=r/d$, $s=q+\delta$, $s'=s/\gcd(3,s)$, $t=q^2+\delta q+1$, $t'=t/d$, $\sigma^s=\rho$, $\tau^t=1$, $\omega=\rho^{(q-1)/d}$. A maximal cyclic subgroup of $\mathrm{SL}(3,q)$ is conjugate to one of the followings: $C_{pr}=\langle\begin{pmatrix}\rho & 1 & \\ & \rho & \\ & & \rho^{-2}\end{pmatrix}\rangle$, $C_r^{(a,b)}=\langle\begin{pmatrix}\rho^a & & \\ & \rho^b & \\ & & \rho^{-a-b}\end{pmatrix}\rangle$ $(0\le a<r'$, $a\le b<r$, $(r,a,b)=1)$, $C_{rs}=\langle\begin{pmatrix}B & \\ & \rho^{-1}\end{pmatrix}\rangle$, ${C_{dp}}^{(c)}=\langle\omega\begin{pmatrix}1 & \theta^c & \\ & 1 & \theta^c \\ & & 1\end{pmatrix}\rangle$ $(0\le c\le d-1)$, C_t, where B is conjugate to $\begin{pmatrix}\sigma^\delta & \\ & \sigma^q\end{pmatrix}$ in $\mathrm{GL}(2,q^2)$ and C_t is generated by an element conjugate to $\begin{pmatrix}\tau & & \\ & \tau^{\delta q} & \\ & & \tau^{q^2}\end{pmatrix}^{q-1}$ in $\mathrm{GL}(3,q^3)$ [4, Table 1a]. Let $\psi\colon \mathrm{SL}(3,q)\to\mathrm{PSL}(3,q)$

be a canonical projection and put $D_{pr'} = \psi(C_{pr})$, $D^{(a,b)}_{r(a,b)} = \psi(C^{(a,b)}_r)$, $D_{r's} = \psi(C_{rs})$, $D^{(c)}_p = \psi(C^{(c)}_{dp})$, $D_{t'} = \psi(C_t)$, where $r(a,b) = r/d$ if $d = 3$ and $ra/d \equiv rb/d \equiv -r(a+b)/d$ modulo r, and $r(a,b) = r$ otherwise. We may assume that $\mathrm{RCycl}(G)$ consists of subgroups of the above cyclic subgroups.

Proposition 15 *If r satisfies the prime condition, then* $\mathrm{PSL}(3,q)$ *is a* CCG.

Proof The order of a maximal cyclic subgroup is r, r', p, $r's$, pr', or t'. We see that $(p, r's) = (pr's, t') = 1$. For $C \in \mathrm{RCycl}_1(G)$, if $D^{(0)}_p \le C$, $D^{(1,1)}_{r'} < C$ or $C < D_{t'}$, then a maximal cyclic subgroup containing C is unique. We see that

$$\begin{aligned}\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) &= \sum_{c=0}^{d-1} \beta_G(D^{(c)}_p)\gamma(D^{(c)}_p) + \beta_G(D_{t'})\gamma(D_{t'}) + \beta_G(D_{r's})\gamma(D_{r's}) \\ &\quad + \beta_G(D_{pr'})\gamma(D_{pr'}) + \sum_{(a,b)} \sum_{C\le D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C).\end{aligned}$$

Note that $\beta_G(D_{pr'}) = 1/r$, $\beta_G(D_{sr'}) = 1/2$, $\beta_G(D_{t'}) = 1/3$ and $\beta_G(D_p) = -s/p^2r$, where D_p is a subgroup of $D_{pr'}$ of order p. Then $\beta_G(D_{pr'}) + \beta_G(D_p) > 0$ and

$$\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \ge \sum_{(a,b)} \sum_{C\le D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C).$$

By the proof of Proposition 10, if r satisfies the prime condition, then for any $D \in \mathrm{RCycl}_1(G)$ of order r, $\sum_C \beta_G(C,D)\gamma(C) \ge 0$ and thus $\sum_{(a,b)} \sum_{C\le D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C) \ge 0$. ∎

Example 16 The number 30 does not satisfy the prime condition. The following table corresponds with $\beta_{\mathrm{PSL}(3,31)}(C,D)$ such that $\beta_{\mathrm{PSL}(3,31)}(C) \ne 0$.

$C\backslash D$	1	2	3	4	$5b$	6	$10a$	$10b$
3	0	0	1/600	0	0	−1/200	0	0
6	0	0	0	0	0	1/100	0	0
$10b$	0	0	0	0	0	0	0	1/29760
$15a$	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$10c$	$10d$	$15a$	$15b$	20	$30a$	$30b$	$30c$	$30d$	$31a$	$31b$
3	0	0	−1/200	−1/200	0	1/200	1/100	1/100	1/200	0	0
6	0	0	0	0	0	−1/100	−1/50	−1/50	−1/100	0	0
$10b$	0	0	0	0	−1/64	0	0	0	−1/60	0	0
$15a$	0	0	1/40	0	0	−1/40	0	−1/20	0	0	0
$15b$	0	0	0	1/40	0	0	−1/20	0	−1/40	0	0
$30a$	0	0	0	0	0	1/20	0	0	0	0	0
$30b$	0	0	0	0	0	0	1/10	0	0	0	0
$30c$	0	0	0	0	0	0	0	1/10	0	0	0
$30d$	0	0	0	0	0	0	0	0	1/20	0	0
$31a$	0	0	0	0	0	0	0	0	0	1/288300	0
$31b$	0	0	0	0	0	0	0	0	0	0	1/930
$31c$	0	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$31c$	$31d$	62	155	310	320	331	$\beta(C)$
3	0	0	0	0	0	0	0	$1/60$
6	0	0	0	0	0	0	0	$-1/20$
$10b$	0	0	0	0	$-1/930$	0	0	$-1/30$
$15a$	0	0	0	0	0	0	0	$-1/20$
$15b$	0	0	0	0	0	0	0	$-1/20$
$30a$	0	0	0	0	0	0	0	$1/20$
$30b$	0	0	0	0	0	0	0	$1/10$
$30c$	0	0	0	0	0	0	0	$1/10$
$30d$	0	0	0	0	0	0	0	$1/20$
$31a$	0	0	$-1/300$	$-1/300$	$1/300$	0	0	$-16/4805$
$31b$	0	0	0	0	0	0	0	$1/930$
$31c$	$1/930$	0	0	0	0	0	0	$1/930$
$31d$	0	$1/930$	0	0	0	0	0	$1/930$
310	0	0	0	0	$1/30$	0	0	$1/30$
320	0	0	0	0	0	$1/2$	0	$1/2$
331	0	0	0	0	0	0	$1/3$	$1/3$

Table 2: $\beta_{\mathrm{PSL}(3,31)}(-,-)$

Since $\beta(6)+\beta(30a)=0$, $\beta(10b)+\beta(30d)>0$, $\beta(15a)+\beta(30c)>0$, $\beta(15b)+\beta(30b)>0$, $\beta(31a)+\beta(310)>0$, we get $g(\mathrm{PSL}(3,31)) \geq \beta(3)\gamma(3)+\beta(31b)\gamma(31b)+\beta(31c)\gamma(31c)+\beta(31d)\gamma(31d)+\beta(320)\gamma(320)+\beta(331)\gamma(331) \geq 0$. Thus $\mathrm{PSL}(3,31)$ is a CCG.

5 Future work

It is not true that for an extension $1 \to H \to G \to K \to 1$, if H and K are CCGs, then G is a CCG.

Proposition 17 *A_{22} is not a* CCG.

Proof Let $x=(1,2,3,4,5,6,7)(8,9,10,11,12)(13,14,15,16)(17,18,19)(20,21) \in A_{22}$ and $S=\{\langle x^3\rangle,\langle x^4\rangle,\langle x^5\rangle,\langle x^7\rangle\}$. Let $\tilde{S}$ be the subset of $\mathrm{RCycl}_1(A_{22})$ consisting of C such that some element of S is subconjugate to C. Then $\tilde{S}=S\cup\{\langle x\rangle,\langle x^2\rangle\}$.

n	1	2	3	4	5	7	sum
$\beta_{A_{22}}(\langle x^n\rangle)$	$\frac{1}{96}$	$-\frac{1}{288}$	$-\frac{1}{384}$	$-\frac{13}{15120}$	$-\frac{7}{3456}$	$-\frac{137}{92160}$	$-\frac{61}{1935360}$

Let $\gamma\colon \mathrm{RCycl}(G) \to \mathbb{Q}_{\geq 0}$ by $\gamma(C)=1$ if C is conjugate to some element of $\tilde{S}$ and $\gamma(C)=0$ otherwise. We see that $\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) = -61/1935360 < 0$. Therefore A_{22} is not a CCG. ∎

Suppose that $\mathrm{RCycl}(A_{22}) \supset \tilde{S}$. There exists representation A_{22}-spaces V and W such that $\dim V^C = \dim W^C$ for $C \in \mathrm{RCycl}(A_{22}) \smallsetminus \tilde{S}$ including $\dim V = \dim W$, and $\dim W^C - \dim V^C$ is constant positive number for $C \in \tilde{S}$. By these condition, we have $\dim W < \dim V$ (see (1)). Suppose that there is an isovariant map $f\colon V \to W$. Let C be a cyclic subgroup generated by x, and H a solvable subgroup of $N_{A_{22}}(C)$ of order 840 generated by x and $(14,16)(18,19)$. Then H normalizes C and $g_f(C) = 1$ and $g_f(H) = \frac{1}{2}$, which is a contradiction. Therefore, there is no isovariant map $V \to W$.

Thus we consider a new condition. For a map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ and a subgroup H of G, we put

$$\hat{\gamma}(H) = \sum_{C \in \mathrm{RCycl}_1(H)} \beta_H(C)\bar{\gamma}(C),$$

where $\bar{\gamma}\colon \mathrm{Cycl}_1(G) \to \mathbb{Q}$ is a class function which sends a cyclic subgroup C of G to $\gamma(C')$ such that $C' \in \mathrm{RCycl}_1(G)$ is conjugate to C in G. If $H \in \mathrm{RCycl}_1(G)$ then $\hat{\gamma}(H) = \gamma(C)$, and if H_1 and H_2 are conjugate in G then $\hat{\gamma}(H_1) = \hat{\gamma}(H_2)$. Recall that $g_f(H_2) - g_f(H_1) = g_{f^{H_1}}(H_2/H_1)$ and if H_2/H_1 is a BUG then $g_{f^{H_1}}(H_2/H_1) \geq 0$ for an isovariant G-map $f\colon V \to W$. A group G is a SCG (subgroup condition group) if for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ such that $\hat{\gamma}(H_1) \leq \hat{\gamma}(H_2)$ for subgroups $H_1 \trianglelefteq H_2 \leq G$ with H_2/H_1 a CCG, $\hat{\gamma}(G) \geq 0$. A SCG is a BUG.

Question 18 *Is the group A_{22} a* SCG*?*

A sufficient condition to be a BUG is that the minimizing value of the following linear programming is zero.

$$\text{Minimize} \sum_{V \in \mathrm{Irr}_1(G)} x_V \dim V$$

$$\text{subject to} \begin{cases} -1 \leq x_V \leq 1, \quad V \in \mathrm{Irr}_1(G) \\ \displaystyle\sum_{V \in \mathrm{Irr}_1(G)} x_V(\dim V^{H_1} - \dim V^{H_2}) \geq 0, \quad H_1 \triangleleft H_2 \leq G, H_2/H_1 \text{ solvable} \end{cases}$$

where $\mathrm{Irr}_1(G)$ is the set of all irreducible nontrivial representation G-spaces.

Since there are many inequalities, we could not check whether the minimizing value is zero or not for A_{22}. We must reduce partial condition to compute.

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